

# PART-WHOLE COMPARISONS OF MEANS



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We consider here the situation in which we have  $m$  randomly drawn observations from population 1 and  $n-m$  observations randomly drawn from population 2, and the data are drawn independently from each of the populations. We designate the mean of the two samples as  $\bar{x}_1$  and  $\bar{x}_2$ , and the sample variances of the two data sets will be designated as  $s_1^2$  and  $s_2^2$ . We also designate by  $\bar{x}_T$  the mean of the total sample of  $n$  observations from the two populations. The object of the t-test is to test whether the mean of population 1 is different from that of the composite population consisting of populations 1 and 2.

In this note we describe the t-test used by The Analytical Group's WinCross (2012) and contrast it with the t-test recommended by the National Assessment of Educational Progress (NAEP) (2009). Finally, in the spirit of the analysis in Eberhardt and Fligner (1977), we compare the two procedures and conclude that the WinCross approach is superior.

## WinCross approach

The hypothesis test is going to be based on the difference between  $\bar{x}_1$  and  $\bar{x}_T$ , and so we need to determine the variance of that difference. Let us assume that the  $n$  observations are ordered so that the first  $m$  are from population 1 and the remaining  $n-m$  are from population 2. Then

$$\begin{aligned} \bar{x}_1 - \bar{x}_T &= \frac{\sum_{i=1}^m x_i}{m} - \frac{\sum_{i=1}^n x_i}{n} \\ &= \frac{\sum_{i=1}^m x_i}{m} - \frac{\sum_{i=1}^m x_i + \sum_{i=m+1}^n x_i}{n} \\ &= \left(\frac{1}{m} - \frac{1}{n}\right) \sum_{i=1}^m x_i - \frac{\sum_{i=m+1}^n x_i}{n} \\ &= \left(1 - \frac{m}{n}\right) \bar{x}_1 - \left(1 - \frac{m}{n}\right) \bar{x}_2 \\ &= \left(1 - \frac{m}{n}\right) (\bar{x}_1 - \bar{x}_2) \end{aligned}$$

Therefore the variance of  $\bar{x}_1 - \bar{x}_T$  is given by

$$\begin{aligned}
V[\bar{x}_1 - \bar{x}_T] &= \left(1 - \frac{m}{n}\right)^2 [V[\bar{x}_1] + V[\bar{x}_2]] \\
&= \left(1 - \frac{m}{n}\right)^2 \left[\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n-m}\right]
\end{aligned}$$

which is estimated by

$$\left(1 - \frac{m}{n}\right)^2 \left[\frac{s_1^2}{m} + \frac{s_2^2}{n-m}\right]$$

That is, if  $p=m/n$  is the fraction of the  $n$  observations that come from population 1, then the variance of  $\bar{x}_1 - \bar{x}_T$  is given by  $(1-p)^2$  times the sum of squares of the standard errors of population 1 and population 2. The  $t$  statistic for testing the hypothesis that the mean of population 1 is different from that of the composite population is given by

$$t = \frac{\bar{x}_1 - \bar{x}_T}{(1-p)\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n-m}}}$$

When the variances of the two populations are identical, the common variance may be estimated by  $s_T^2$  and so this reduces to

$$\begin{aligned}
t &= \frac{\bar{x}_1 - \bar{x}_T}{(1-p)\sqrt{\frac{s_T^2}{m} + \frac{s_T^2}{n-m}}} \\
&= \frac{\bar{x}_1 - \bar{x}_T}{s_T(1-p)\sqrt{\frac{1}{m} + \frac{1}{n-m}}}
\end{aligned}$$

The denominator of  $t$  can be rewritten as

$$s_T(1-p)\sqrt{\frac{1}{np} + \frac{1}{n(1-p)}} = \frac{s_T}{\sqrt{n}}\sqrt{\frac{1-p}{p}}$$

As  $p$  gets close to 1 the denominator of  $t$  approaches 0, so that  $t$  approaches infinity. In this case the test will find significant differences whenever the “part” is almost equal to the “whole.” As a precaution, I recommend not performing this test when  $p > 0.9$ . As  $p$  gets close to 0 the denominator of  $t$  approaches infinity, so that  $t$  approaches 0. In this case the test will find no significant differences whenever the “part” is an infinitesimal part of the “whole.” As a precaution, I recommend not performing this test when  $p < 0.1$ .

### NAEP approach

Alternatively,

$$V[\bar{x}_1 - \bar{x}_T] = V[\bar{x}_1] + V[\bar{x}_T] - 2C[\bar{x}_1, \bar{x}_T]$$

and so we must determine  $C[\bar{x}_1, \bar{x}_T]$ , the covariance of  $\bar{x}_1$  and  $\bar{x}_T$ . This covariance is

$$\begin{aligned}
C[\bar{x}_1, \bar{x}_T] &= C\left[\frac{\sum_{i=1}^m x_i}{m}, \frac{\sum_{i=1}^n x_i}{n}\right] \\
&= \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n C[x_i, x_j] \\
&= \frac{1}{mn} m\sigma_1^2
\end{aligned}$$

and so

$$\begin{aligned}
V[\bar{x}_1 - \bar{x}_T] &= V[\bar{x}_1] + V[\bar{x}_T] - 2C[\bar{x}_1, \bar{x}_T] \\
&= \frac{\sigma_1^2}{m} + \frac{\sigma_T^2}{n} - 2\frac{\sigma_1^2}{n} \\
&= \frac{\sigma_T^2}{n} + \frac{\sigma_1^2}{m}(1 - 2p)
\end{aligned}$$

which is the expression given by NAEP.

Note that when  $m=n/2$  this variance reduces to that of  $\bar{x}_T$ . (Note also that when  $m=n$  this reduces to 0, as it should, because  $\bar{x}_T$  is identical to  $\bar{x}_1$ .)

### Comparison

A way of seeing the equivalence of these expressions for the variance of the difference of the part and whole means is by expressing  $\bar{x}_T$  as

$$\bar{x}_T = \frac{\sum_{i=1}^n x_i}{n} = \frac{m\bar{x}_1 + (n-m)\bar{x}_2}{n} = p\bar{x}_1 + (1-p)\bar{x}_2$$

where  $p=m/n$ . Then

$$V\bar{x}_T = \frac{\sigma_T^2}{n} = p^2 \frac{\sigma_1^2}{m} + (1-p)^2 \frac{\sigma_2^2}{n-m}$$

and so

$$\begin{aligned}
V[\bar{x}_1 - \bar{x}_T] &= \frac{\sigma_T^2}{n} + \frac{\sigma_1^2}{m}(1 - 2p) \\
&= p^2 \frac{\sigma_1^2}{m} + (1-p)^2 \frac{\sigma_2^2}{n-m} + \frac{\sigma_1^2}{m}(1 - 2p) \\
&= (1-p)^2 \frac{\sigma_1^2}{m} + (1-p)^2 \frac{\sigma_2^2}{n-m} \\
&= (1-p)^2 \left[ \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n-m} \right]
\end{aligned}$$

Unfortunately, though in the population

$$\frac{\sigma_T^2}{n} + \frac{\sigma_1^2}{m}(1 - 2p) = (1-p)^2 \left[ \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n-m} \right]$$

this equality does not hold for the sample counterparts:

$$d_1 = \frac{s_T^2}{n} + \frac{s_1^2}{m}(1-2p), d_2 = (1-p)^2 \left[ \frac{s_1^2}{m} + \frac{s_2^2}{n-m} \right]$$

To see the relationship between these sample variances, we note that

$$\begin{aligned} (n-1)s_T^2 &= \sum_{i=1}^n (x_i - \bar{x}_T)^2 = \sum_{i=1}^m (x_i - \bar{x}_1 + \bar{x}_1 - \bar{x}_T)^2 + \sum_{i=m+1}^n (x_i - \bar{x}_2 + \bar{x}_2 - \bar{x}_T)^2 \\ &= \sum_{i=1}^m (x_i - \bar{x}_1)^2 + m(\bar{x}_1 - \bar{x}_T)^2 + \sum_{i=m+1}^n (x_i - \bar{x}_2)^2 + (n-m)(\bar{x}_2 - \bar{x}_T)^2 \\ &= (m-1)s_1^2 + (n-m-1)s_2^2 + m(\bar{x}_1 - \bar{x}_T)^2 + (n-m)(\bar{x}_2 - \bar{x}_T)^2 \end{aligned}$$

Then

$$d_1 = \frac{(m-1)s_1^2}{n(n-1)} + \frac{(n-m-1)s_2^2}{n(n-1)} + \frac{m}{n(n-1)}(\bar{x}_1 - \bar{x}_T)^2 + \frac{n-m}{n(n-1)}(\bar{x}_2 - \bar{x}_T)^2 + \frac{s_1^2}{m}(1-2p)$$

As we let m and n approach infinity but with m/n=p, nd<sub>1</sub> approaches

$$\begin{aligned} p\sigma_1^2 + (1-p)\sigma_2^2 + p(\mu_1 - \mu_T)^2 + (1-p)(\mu_2 - \mu_T)^2 + (1/p-2)\sigma_1^2 \\ = \frac{1-p^2}{p}\sigma_1^2 + (1-p)\sigma_2^2 + p(\mu_1 - \mu_T)^2 + (1-p)(\mu_2 - \mu_T)^2 \end{aligned}$$

and nd<sub>2</sub> approaches

$$(1-p)^2 \left[ \frac{\sigma_1^2}{p} + \frac{\sigma_2^2}{1-p} \right] = \frac{(1-p)^2}{p}\sigma_1^2 + (1-p)\sigma_2^2$$

Under the null hypothesis the terms in nd<sub>1</sub> involving the means are both equal to 0, leaving us with a comparison of

$$\frac{1-p^2}{p}\sigma_1^2 + (1-p)\sigma_2^2$$

and

$$\frac{(1-p)^2}{p}\sigma_1^2 + (1-p)\sigma_2^2$$

Since  $1-p^2 > (1-p)^2$ , we see that even when the null hypothesis is true d<sub>1</sub> is larger than d<sub>2</sub>. And when the null hypothesis is false d<sub>1</sub> is much greater than d<sub>2</sub> because of the added terms involving the means.

In summary, the NAEP formula is based on a larger standard error for the difference between the part and whole mean than that used by WinCross, thereby leading to fewer significant differences detected than are actually to be found in the data.

## References

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