**WinCross's Sample Balancing (Goodness-of-fit minimization technique) Program**

WinCross's Sample Balancing Program determines weights to be applied to each respondent in a sample in order that the weighted marginal distributions on each of v variables match preset distributions on those v variables. The program is an adaptation of the procedure known today as "iterative proportional fitting" and referred to by some as "rim weighting." The original iterative proportional fitting procedure was devised by W. Edwards Deming and Frederick F. Stephan, first published in their December, 1940 paper, "On a Least Squares Adjustment of a Sampled Frequency Table when the Expected Marginal Totals are Known," in Volume 11 of *The Annals of Mathematical Statistics*, pages 427-444, and further explicated in Chapter 7 of Deming's book, Statistical Adjustment of Data (New York: John Wiley & Sons, 1943). WinCross's adaptation was developed by J. Stephens Stock, a colleague of Deming, in the 1960s.

To describe the procedure used by the program, we must first establish some notation. Let v be the number of variables to be considered in the balancing. Let $c_i$ denote the number of levels (sometimes referred to as "breaks") of the i-th variable, i=1,…,v. Let $p_{j_1 \cdots j_v}$ denote the proportion of respondents in the sample in level $j_1$ on variable 1, $j_2$ on variable 2, …, $j_v$ on variable v, where $j_i = 1, \ldots, c_i$. Let $f_{j_i}^i$ denote the marginal proportion in the sample of level $j_i$ of variable i ($j_i=1,\ldots,c_i$, $i=1,\ldots,v$).

To make things concrete, let v=3, with the three variables being income (i=1), age (i=2), and region (i=3). Suppose there are 5 income breaks ($c_1=5$), 10 age breaks ($c_2=10$), and 9 region breaks ($c_3=9$). Then, in our notation, if for example $j_1=2$, $j_2=1$, and $j_3=4$, then $p_{j_1 j_2 j_3} = p_{214}$ is the proportion of the sample that are of income level 2, age level 1, and region level 4. And, as another example of the interpretation of this notation, if i=3 then $f_{j_i}^i = f_{j_3}^3 = f_2^3$ is the proportion of the sample that are in region level 2 (the superscript "3" indicates that we are looking at variable 3, region, and the subscript "2" indicates that we are looking at level 2 of that variable).

Clearly $f_{j_i}^i$ can be determined by adding up all the $p_{j_1 \cdots j_v}$ across all the values of each of the v-1 $j_k$ for which $k \neq i$. For example, to obtain $f_2^3$ one adds up all the proportions $p_{j_1 j_2 2}$ across $j_1=1,2$ and $j_2=1,2,3$. We express this relation symbolically as

$$f_{j_i}^i = \sum_{j_k, k \neq i} p_{j_1 \cdots j_v}$$

These $f_{j_i}^i$ are called sample rim percents.

Suppose that the preset distributions on the v variables are given by the set of target proportions $g_{j_i}^i$. The object of the sample balancing program is to find a set of weights $w_{j_1 \cdots j_v}$ such that if, when looking at the $j_i$-th break, instead of adding up the

$p_{j_1 \cdots j_v}$ across all but the i-th category, we add up the $w_{j_1 \cdots j_v} p_{j_1 \cdots j_v}$ across all but the i-th category, we will obtain the $g_{j_i}^i$. That is,

$$g_{j_i}^i = \sum_{j_k, k \neq i} w_{j_1 \cdots j_v} p_{j_1 \cdots j_v}$$

To illustrate the WinCross sample balancing program, we will use a data set of 1000 respondents, whose income, age, and region distribution are given in the table below, along with the target percentages input into the sample balancing program.

| | | n | sample f rim percents | target g rim percents |
|---|---|---|---|---|
| Income i=1 | | | | |
| | 1 | 124 | 12.40 | 17.95 |
| | 2 | 150 | 15.00 | 23.20 |
| | 3 | 305 | 30.50 | 27.28 |
| | 4 | 221 | 22.10 | 14.34 |
| | 5 | 200 | 20.00 | 17.23 |
| Total | | 1000 | 100.00 | 100.00 |
| Age i=2 | | | | |
| | 1 | 6 | 0.60 | 3.87 |
| | 2 | 34 | 3.40 | 9.07 |
| | 3 | 185 | 18.50 | 19.09 |
| | 4 | 276 | 27.60 | 21.60 |
| | 5 | 248 | 24.80 | 18.02 |
| | 6 | 92 | 9.20 | 6.44 |
| | 7 | 64 | 6.40 | 5.17 |
| | 8 | 65 | 6.50 | 8.80 |
| | 9 | 26 | 2.60 | 5.91 |
| | 10 | 4 | 0.40 | 2.03 |
| Total | | 1000 | 100.00 | 100.00 |
| Region i=3 | | | | |
| | 1 | 55 | 5.50 | 5.14 |
| | 2 | 79 | 7.90 | 14.21 |
| | 3 | 187 | 18.70 | 16.44 |
| | 4 | 129 | 12.90 | 7.14 |
| | 5 | 169 | 16.90 | 19.05 |
| | 6 | 56 | 5.60 | 6.30 |
| | 7 | 103 | 10.30 | 10.91 |
| | 8 | 82 | 8.20 | 6.40 |
| | 9 | 140 | 14.00 | 14.41 |
| Total | | 1000 | 100.00 | 100.00 |

In this example there are three *variables* (income, age, and region) to be used in determining the weights, with 5, 10, and 9 *levels*, respectively. There are potentially 5 x 10 x 9 = 450 cells in the three-way crosstabulation of these data. In actuality, there are only 275 cells for which there is at least one respondent among the 1000 in the sample. The frequency distribution across these 275 cells is given in Appendix I.

The procedure for determining the weights is iterative. Each iterative "round" consists of v "passes," one "pass" through each of the v variables. We begin at "round 0" by setting all weights $w_{j_1} \cdots_{j_v}(0,i)$ equal to 1, i.e., we begin with the unweighted data.

Suppose we are on the i-th "pass" in "round t+1." Let $w_{j_1} \cdots_{j_v}(t,i)$ denote the weights at this point in the iterative process. Let $g^i_{j_i}(t)$ denote the results of the computation

$$g^i_{j_i}(t) = \sum_{j_k, k \neq i} w_{j_1} \cdots_{j_v}(t,i) p_{j_1} \cdots_{j_v}$$

These $g^i_{j_i}(t)$ are called <u>estimated target rim percents</u>.

At the first pass (i=1) of the t-th round of the iterative procedure the program calculates a set of increments $d_{j_1} \cdots_{j_v}(t,1)$ to add to the $w_{j_1} \cdots_{j_v}(t-1,v)$, producing $w_{j_1} \cdots_{j_v}(t,1) = w_{j_1} \cdots_{j_v}(t-1,v) + d_{j_1} \cdots_{j_v}(t,1)$. At the i-th pass (i>1) of the t-th round of the iterative procedure the program calculates a set of increments $d_{j_1} \cdots_{j_v}(t,i)$ to add to the $w_{j_1} \cdots_{j_v}(t,i-1)$, producing $w_{j_1} \cdots_{j_v}(t,i) = w_{j_1} \cdots_{j_v}(t,i-1) + d_{j_1} \cdots_{j_v}(t,i)$.

These increments are given by the formula

$$d_{j_1 j_2 \cdots j_v} = [g^i_{j_i}(t) - g^i_{j_i}] / f^i_{j_i}$$

That is, we compare the ratio of the estimated target rim percent to the sample rim percent to the ratio of the target rim percent to the sample rim percent, and increment or decrement by the difference between these two ratios.

Let us illustrate this on our sample data. The first pass of round 1 considers the first variable, income. Suppose one wanted to only balance to the marginal on ***income***. The way to determine those weights is a simple process: weight each respondent by a factor such that the sum of the weights equals the target for each level. The appropriate weights that accomplish this are simply the ratio of the target rim percents to the sample rim percents for each level of income. This is illustrated in the following table (since the total count is 1000, we multiply our sample and target percents by 10 I what follows):

| income | sample | target | target/sample |
|---|---|---|---|
| 1 | 124 | 179.5 | 1.44758 |
| 2 | 150 | 232.0 | 1.54667 |
| 3 | 305 | 272.8 | 0.89443 |
| 4 | 221 | 143.4 | 0.64887 |
| 5 | 200 | 172.3 | 0.86150 |
| | 1000 | 1000.0 | |

When one applies the income weight (in this case, the initial weight 1.0) to each respondent and looks at the weighted marginals by income, one obtains the following table:

| income | sample | est target | est target/sample |
|---|---|---|---|
| 1 | 124 | 124 | 1.0 |
| 2 | 150 | 150 | 1.0 |
| 3 | 305 | 305 | 1.0 |
| 4 | 221 | 221 | 1.0 |
| 5 | 200 | 200 | 1.0 |
| | 1000 | 1000 | |

Our goal is to achieve the target, and so we compare the ratio of estimated target rim percent to sample rim percent with the ratio of target rim percent to sample rim percent, and calculate the additive or subtractive adjustment to the initial weight of 1.0, labeled "delta", necessary to make these two ratios equal. The following table presents this comparison and adjustment.

| income | target/sample | est target /sample | delta |
|---|---|---|---|
| 1 | 1.44758 | 1.0 | 0.44758 |
| 2 | 1.54667 | 1.0 | 0.54667 |
| 3 | 0.89443 | 1.0 | -0.10557 |
| 4 | 0.64887 | 1.0 | -0.35113 |
| 5 | 0.86150 | 1.0 | -0.13850 |

So, for example, the new weight for income group 1 is 1.0 + 0.44758=1.44758.

We now move to pass 2 of round 1, where we look at the same initial table for the next variable, ***age.***

| age | sample | target | target/sample |
|---|---|---|---|
| 1 | 6 | 38.7 | 6.45000 |
| 2 | 34 | 90.7 | 2.66765 |
| 3 | 185 | 190.9 | 1.03189 |
| 4 | 276 | 216.0 | 0.78261 |
| 5 | 248 | 180.2 | 0.72661 |
| 6 | 92 | 64.4 | 0.70000 |
| 7 | 64 | 51.7 | 0.80781 |
| 8 | 65 | 88.0 | 1.35385 |
| 9 | 26 | 59.1 | 2.27308 |
| 10 | 4 | 20.3 | 5.07500 |
| | 1000 | 1000.0 | |

When one applies the income weight to each respondent and looks at the weighted marginals by age, one obtains the following table:

| age | sample | est target | est target/sample |
|---|---|---|---|
| 1 | 6 | 5.58 | 0.92923 |
| 2 | 34 | 41.17 | 1.21103 |
| 3 | 185 | 182.08 | 0.98421 |
| 4 | 276 | 277.32 | 1.00478 |
| 5 | 248 | 237.35 | 0.95708 |
| 6 | 92 | 91.18 | 0.99111 |
| 7 | 64 | 63.03 | 0.98482 |
| 8 | 65 | 67.97 | 1.04564 |
| 9 | 26 | 30.12 | 1.15848 |
| 10 | 4 | 4.20 | 1.04925 |
| | 1000 | 1000.00 | |

As our goal is to achieve the target, we compare the ratio of weighted data to data with the ratio of target to data, and calculate the additive or subtractive adjustment to the current respondent weight, labeled "delta", necessary to make these two ratios equal. The following table presents this comparison and adjustment.

| age | target/sample | est target/sample | delta |
|---|---|---|---|
| 1 | 6.45000 | 0.92923 | 5.52077 |
| 2 | 2.66765 | 1.21103 | 1.45662 |
| 3 | 1.03189 | 0.98421 | 0.04768 |
| 4 | 0.78261 | 1.00478 | -0.22218 |
| 5 | 0.72661 | 0.95708 | -0.23046 |
| 6 | 0.70000 | 0.99111 | -0.29111 |
| 7 | 0.80781 | 0.98482 | -0.17701 |
| 8 | 1.35385 | 1.04564 | 0.30821 |
| 9 | 2.27308 | 1.15848 | 1.11460 |
| 10 | 5.07500 | 1.04925 | 4.02575 |

How are these deltas used? Let's look at a few examples. Consider a respondent whose income group was 2. After the first pass of round 1 he would be given a weight of 1.54667. Now suppose his age group were 10. After the second pass of round 1 his weight would be 1.54667+4.02575=5.57242. And if his age group were 4 his weight would now be 1.54667-0.22218=1.32449.

We now move to pass 3 of round 1, where we consider the next variable, ***region.*** We first develop our goal, the "target/sample" ratio, given in the following table.

| region | sample | target | target/sample |
|---|---|---|---|
| 1 | 55 | 51.4 | 0.93455 |
| 2 | 79 | 142.1 | 1.79873 |
| 3 | 187 | 164.4 | 0.87914 |
| 4 | 129 | 71.4 | 0.55349 |
| 5 | 169 | 190.5 | 1.12722 |
| 6 | 56 | 63.0 | 1.12500 |
| 7 | 103 | 109.1 | 1.05922 |
| 8 | 82 | 64.0 | 0.78049 |
| 9 | 140 | 144.1 | 1.02929 |
|  | 1000 | 1000.0 |  |

When one applies the weight based on both income and age to each respondent and looks at the weighted marginals by region, one obtains the following table:

| region | sample | est target | est target/sample |
|---|---|---|---|
| 1 | 55 | 49.27 | 0.89579 |
| 2 | 79 | 81.62 | 1.03317 |
| 3 | 187 | 184.37 | 0.98593 |
| 4 | 129 | 143.46 | 1.11207 |
| 5 | 169 | 168.87 | 0.99920 |
| 6 | 56 | 58.81 | 1.05024 |
| 7 | 103 | 104.50 | 1.01457 |
| 8 | 82 | 74.85 | 0.91278 |
| 9 | 140 | 134.26 | 0.95899 |
| | 1000 | 1000.00 | |

Once again our goal is to achieve the target, and so we compare the ratio of weighted data to data with the ratio of target to data, and calculate the additive or subtractive adjustment to each respondent's weight, labeled "delta", necessary to make these two ratios equal. The following table presents this comparison and adjustment.

| region | target/sample | est target/sample | delta |
|---|---|---|---|
| 1 | 0.93455 | 0.89579 | 0.03876 |
| 2 | 1.79873 | 1.03317 | 0.76557 |
| 3 | 0.87914 | 0.98593 | -0.10679 |
| 4 | 0.55349 | 1.11207 | -0.55858 |
| 5 | 1.12722 | 0.99920 | 0.12802 |
| 6 | 1.12500 | 1.05024 | 0.07476 |
| 7 | 1.05922 | 1.01457 | 0.04465 |
| 8 | 0.78049 | 0.91278 | -0.13229 |
| 9 | 1.02929 | 0.95899 | 0.07030 |

Continuing with our examples, consider a respondent whose income group was 2 and his age group was 10. After pass 2 of round 1 he would be given a weight of 5.57242. If his region were 2 his new weight after pass 3 of round 1 would be 5.57242+0.76557=6.33799. And if his region were 4 his weight would now be 5.57242-0.55858=5.01383.

The WinCross sample balancing program now applies these new weights to the respondents and begins round 2, once again in pass 1 looking at the income marginals. The principle in each step is the same: adjust the weights so that the ratio of estimated target rim percents data to sample rim percents equals the ratio of target rim percents to sample rim percents.

The program continues iterating until a criterion of goodness of fit has been met. WinCross uses the measure

$$\sqrt{\sum_{i=1}^{v}\sum_{j_i=1}^{c_i}[(g_{j_i}^{i}(t)-g_{j_i}^{i})\,/\,f_{j_i}^{i}]^2\,/\,m}$$

where

$$m=\sum_{i=1}^{v}c_i$$

is the total number of levels in the balancing process. (The reason for dividing by m is so that the measure of goodness of fit will have the same scale, regardless of the number of levels being balanced.)

The square of this measure is the average across levels and variables of the sum of squares of deviations between the ratio of estimated target rim percents to sample rim percents and the ratio of actual target rim percents to sample rim percents. The module iterates until this measure is less than some preset value (with default set at 0.00005). This is in contrast to iterative proportional fitting, which has no overall criterion and iterates until each $g_{j_i}^{i}(t)$ is within some preset distance from $g_{j_i}^{i}$, that is, until each estimated target rim percent is within some preset distance from the actual target rim percent.

Appendix I also contains the results of the application of the WinCross sample balancing program to these data. The program iterations ended after the fourth pass. Appendix I contains the resulting weights for each of the 275 combinations of responses found in the data, as well as the total weight for all the respondents with that given combination. For example, the 11 respondents who were in income group 5, age group 4, and region 5 each are weighted by the factor 0.76018.

Here is the buildup of that weight from the four rounds (with three passes per round) of the WinCross sample balancing. The "weight" is the one derived above for the appropriate variable level and the "cum weight" is the sum of the "delta" and the previous "cum weight" (except for the first step, where the previous "cum weight" is implicitly 1, so that the first "delta" is implicitly -0.13850.)

| | delta | cum weight |
|---|---|---|
| income | -0.13850 | 0.86150 |
| age | -0.22218 | 0.63932 |
| region | 0.12802 | 0.76734 |
| income | -0.01178 | 0.75556 |
| age | -0.00391 | 0.75165 |
| region | 0.00694 | 0.75859 |
| income | 0.00123 | 0.75982 |
| age | -0.00024 | 0.75958 |
| region | 0.00044 | 0.76002 |
| income | 0.00016 | 0.76018 |
| age | -0.00003 | 0.76015 |

| | | |
|---|---|---|
| region | 0.00002 | 0.76017 |

One situation that might occur in using the WinCross sample balancing program is that it will produce "negative" weights. The following table presents the buildup of the weight for the respondents with in (income group 4, age group 5, region 4), (income group 4, age group 4, region 4), and (income group 5, age group 6, region 4).

| | 454 | | 444 | | 564 | |
|---|---|---|---|---|---|---|
| | delta | cum weight | delta | cum weight | delta | cum weight |
| income | -0.35113 | 0.64887 | -0.35113 | 0.64887 | -0.13850 | 0.86150 |
| age | -0.23046 | 0.41841 | -0.22218 | 0.42669 | -0.29111 | 0.57039 |
| region | -0.55858 | **-0.14018** | -0.55858 | **-0.13189** | -0.55858 | 0.01181 |
| income | 0.06625 | -0.07392 | 0.06625 | -0.06564 | -0.01178 | 0.00003 |
| age | -0.02374 | -0.09767 | -0.00391 | -0.06954 | -0.01621 | **-0.01618** |
| region | 0.00000 | -0.09767 | 0.00000 | -0.06955 | 0.00000 | -0.01619 |
| income | 0.00393 | -0.09374 | 0.00393 | -0.06562 | 0.00122 | -0.01496 |
| age | -0.00084 | -0.09458 | -0.00024 | -0.06586 | -0.00055 | -0.01551 |
| region | 0.00041 | -0.09417 | 0.00041 | -0.06545 | 0.00041 | -0.01510 |
| income | 0.00020 | -0.09397 | 0.00020 | -0.06525 | 0.00016 | -0.01494 |
| age | -0.00007 | -0.09404 | -0.00003 | -0.06528 | -0.00001 | -0.01495 |
| region | 0.00003 | -0.09401 | 0.00003 | -0.06525 | 0.00003 | -0.01492 |

Note that, in particular due to the first round adjustment for region 4, the cumulative weights become negative. This is due to the extreme alteration that is needed for region 4, where there were 129 respondents and the target is 71.4. The WinCross sample balancing algorithm makes a major reduction in the weight in the first round, so much so that it may produce negative weights. One way to avoid this is to replace those weights with a small number, such as 0.001, as is done in Appendix I.

How does the WinCross sample balancing program compare with iterative proportional fitting? Following are the estimated target rim percents produced by both programs (we used the Quantum implementation of iterative proportional fitting as our comparison). The rightmost two columns contain the elements of the criterion measure used in the WinCross sample balancing program, i.e., the values of the squares of the

$$d_{j_1 j_2 \cdots j_v} = [g^i_{j_i}(t) - g^i_{j_i}] / f^i_{j_i}$$

for each level of each variable.

| | data | target | WinCross | IPF | WinCross | IPF |
|---|---|---|---|---|---|---|
| income | | | | | | |
| 1 | 124 | 179.5 | 179.4892 | 179.5003 | 0.000000007586 | 0.000000000006 |
| 2 | 150 | 232.0 | 231.9841 | 231.9992 | 0.000000011236 | 0.000000000028 |
| 3 | 305 | 272.8 | 272.7745 | 272.7999 | 0.000000006990 | 0.000000000000 |
| 4 | 221 | 143.4 | 143.3749 | 143.3990 | 0.000000012899 | 0.000000000020 |
| 5 | 200 | 172.3 | 172.3763 | 172.2998 | 0.000000145542 | 0.000000000001 |
| Total | 1000 | 1000.0 | 999.9990 | 999.9982 | 0.000000000001 | 0.000000000003 |
| age | | | | | | |
| 1 | 6 | 38.7 | 38.6998 | 38.6668 | 0.000000001111 | 0.000030617778 |
| 2 | 34 | 90.7 | 90.6999 | 90.7535 | 0.000000000009 | 0.000002475995 |
| 3 | 185 | 190.9 | 190.8987 | 190.9080 | 0.000000000049 | 0.000000001870 |
| 4 | 276 | 216.0 | 216.0004 | 215.9743 | 0.000000000002 | 0.000000008671 |
| 5 | 248 | 180.2 | 180.2011 | 180.1599 | 0.000000000020 | 0.000000026145 |
| 6 | 92 | 64.4 | 64.3999 | 64.3959 | 0.000000000001 | 0.000000001986 |
| 7 | 64 | 51.7 | 51.6993 | 51.6988 | 0.000000000120 | 0.000000000352 |
| 8 | 65 | 88.0 | 88.0001 | 88.0201 | 0.000000000002 | 0.000000095624 |
| 9 | 26 | 59.1 | 59.0998 | 59.1316 | 0.000000000059 | 0.000001477160 |
| 10 | 4 | 20.3 | 20.2999 | 20.2893 | 0.000000000625 | 0.000007155625 |
| Total | 1000 | 1000.0 | 999.9989 | 999.9982 | 0.000000000001 | 0.000000000003 |
| region | | | | | | |
| 1 | 55 | 51.4 | 51.3999 | 51.3596 | 0.000000000003 | 0.000000539557 |
| 2 | 79 | 142.1 | 142.0999 | 142.2019 | 0.000000000002 | 0.000001663773 |
| 3 | 187 | 164.4 | 164.3999 | 164.4212 | 0.000000000000 | 0.000000012853 |
| 4 | 129 | 71.4 | 71.3999 | 71.4617 | 0.000000000001 | 0.000000228766 |
| 5 | 169 | 190.5 | 190.4999 | 190.5803 | 0.000000000000 | 0.000000225766 |
| 6 | 56 | 63.0 | 62.9999 | 62.9725 | 0.000000000003 | 0.000000241151 |
| 7 | 103 | 109.1 | 109.0999 | 109.0077 | 0.000000000001 | 0.000000803025 |
| 8 | 82 | 64.0 | 63.9999 | 63.9870 | 0.000000000001 | 0.000000025134 |
| 9 | 140 | 144.1 | 144.0999 | 144.0063 | 0.000000000001 | 0.000000447943 |
| Total | 1000 | 1000.0 | 999.9991 | 999.9982 | | |

The criterion measure used by WinCross is the square root of the sum of the values in the right hand columns of the above tables, divided by the square root of m=24, the number of breaks in the data. For WinCross this value is 0.000431586/4.898979=0.000088097; for iterative proportional fitting this value is 0.006785959/4.898979=.00138518, over 15 times as large.

APPENDIX I

| combination | income | age | region | count | weight | wtd count |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 5 | 1 | 7.03868 | 7.03868 |
| 2 | 1 | 2 | 3 | 5 | 2.64602 | 13.23010 |
| 3 | 1 | 2 | 5 | 1 | 2.89112 | 2.89112 |
| 4 | 1 | 2 | 6 | 1 | 2.83734 | 2.83734 |
| 5 | 1 | 2 | 7 | 1 | 2.80156 | 2.80156 |
| 6 | 1 | 3 | 1 | 1 | 1.44292 | 1.44292 |
| 7 | 1 | 3 | 3 | 3 | 1.29764 | 3.89292 |
| 8 | 1 | 3 | 4 | 4 | 0.84916 | 3.39664 |
| 9 | 1 | 3 | 5 | 1 | 1.54273 | 1.54273 |
| 10 | 1 | 3 | 6 | 2 | 1.48895 | 2.97790 |
| 11 | 1 | 3 | 7 | 3 | 1.45317 | 4.35951 |
| 12 | 1 | 3 | 8 | 2 | 1.27067 | 2.54134 |
| 13 | 1 | 3 | 9 | 1 | 1.46797 | 1.46797 |
| 14 | 1 | 4 | 1 | 2 | 1.13520 | 2.27040 |
| 15 | 1 | 4 | 2 | 2 | 1.87272 | 3.74544 |
| 16 | 1 | 4 | 3 | 3 | 0.98992 | 2.96976 |
| 17 | 1 | 4 | 4 | 7 | 0.54144 | 3.79008 |
| 18 | 1 | 4 | 5 | 11 | 1.23501 | 13.58511 |
| 19 | 1 | 4 | 6 | 2 | 1.18123 | 2.36246 |
| 20 | 1 | 4 | 7 | 4 | 1.14545 | 4.58180 |
| 21 | 1 | 4 | 8 | 4 | 0.96296 | 3.85184 |
| 22 | 1 | 4 | 9 | 5 | 1.16026 | 5.80130 |
| 23 | 1 | 5 | 1 | 2 | 1.10644 | 2.21288 |
| 24 | 1 | 5 | 2 | 3 | 1.84395 | 5.53185 |
| 25 | 1 | 5 | 3 | 4 | 0.96116 | 3.84464 |
| 26 | 1 | 5 | 4 | 2 | 0.51268 | 1.02536 |
| 27 | 1 | 5 | 5 | 5 | 1.20625 | 6.03125 |
| 28 | 1 | 5 | 6 | 1 | 1.15247 | 1.15247 |
| 29 | 1 | 5 | 8 | 2 | 0.93420 | 1.86840 |
| 30 | 1 | 5 | 9 | 1 | 1.13149 | 1.13149 |
| 31 | 1 | 6 | 1 | 1 | 1.05367 | 1.05367 |
| 32 | 1 | 6 | 2 | 1 | 1.79119 | 1.79119 |
| 33 | 1 | 6 | 3 | 1 | 0.90839 | 0.90839 |
| 34 | 1 | 6 | 4 | 2 | 0.45992 | 0.91984 |
| 35 | 1 | 6 | 5 | 2 | 1.15349 | 2.30698 |
| 36 | 1 | 6 | 6 | 1 | 1.09971 | 1.09971 |
| 37 | 1 | 6 | 8 | 1 | 0.88143 | 0.88143 |
| 38 | 1 | 6 | 9 | 1 | 1.07873 | 1.07873 |
| 39 | 1 | 7 | 1 | 1 | 1.20926 | 1.20926 |
| 40 | 1 | 7 | 2 | 1 | 1.94677 | 1.94677 |
| 41 | 1 | 7 | 5 | 2 | 1.30907 | 2.61814 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 42 | 1 | 7 | 6 | 1 | 1.25529 | 1.25529 |
| 43 | 1 | 7 | 7 | 1 | 1.21951 | 1.21951 |
| 44 | 1 | 7 | 9 | 1 | 1.23431 | 1.23431 |
| 45 | 1 | 8 | 2 | 4 | 2.38434 | 9.53736 |
| 46 | 1 | 8 | 3 | 2 | 1.50154 | 3.00308 |
| 47 | 1 | 8 | 4 | 3 | 1.05307 | 3.15921 |
| 48 | 1 | 8 | 5 | 1 | 1.74664 | 1.74664 |
| 49 | 1 | 8 | 9 | 1 | 1.67188 | 1.67188 |
| 50 | 1 | 9 | 3 | 3 | 2.45789 | 7.37367 |
| 51 | 1 | 9 | 4 | 2 | 2.00941 | 4.01882 |
| 52 | 1 | 9 | 5 | 2 | 2.70298 | 5.40596 |
| 53 | 1 | 9 | 9 | 3 | 2.62822 | 7.88466 |
| 54 | 2 | 2 | 3 | 1 | 2.84731 | 2.84731 |
| 55 | 2 | 2 | 4 | 2 | 2.39883 | 4.79766 |
| 56 | 2 | 2 | 5 | 3 | 3.09240 | 9.27720 |
| 57 | 2 | 2 | 7 | 4 | 3.00284 | 12.01136 |
| 58 | 2 | 2 | 8 | 1 | 2.82035 | 2.82035 |
| 59 | 2 | 3 | 3 | 4 | 1.49892 | 5.99568 |
| 60 | 2 | 3 | 4 | 4 | 1.05045 | 4.20180 |
| 61 | 2 | 3 | 5 | 4 | 1.74402 | 6.97608 |
| 62 | 2 | 3 | 6 | 4 | 1.69024 | 6.76096 |
| 63 | 2 | 3 | 7 | 3 | 1.65446 | 4.96338 |
| 64 | 2 | 3 | 8 | 4 | 1.47196 | 5.88784 |
| 65 | 2 | 3 | 9 | 5 | 1.66926 | 8.34630 |
| 66 | 2 | 4 | 1 | 3 | 1.33649 | 4.00947 |
| 67 | 2 | 4 | 2 | 6 | 2.07401 | 12.44406 |
| 68 | 2 | 4 | 3 | 8 | 1.19121 | 9.52968 |
| 69 | 2 | 4 | 4 | 6 | 0.74273 | 4.45638 |
| 70 | 2 | 4 | 5 | 6 | 1.43630 | 8.61780 |
| 71 | 2 | 4 | 6 | 2 | 1.38252 | 2.76504 |
| 72 | 2 | 4 | 7 | 4 | 1.34674 | 5.38696 |
| 73 | 2 | 4 | 8 | 2 | 1.16425 | 2.32850 |
| 74 | 2 | 4 | 9 | 3 | 1.36154 | 4.08462 |
| 75 | 2 | 5 | 1 | 1 | 1.30773 | 1.30773 |
| 76 | 2 | 5 | 2 | 1 | 2.04524 | 2.04524 |
| 77 | 2 | 5 | 3 | 6 | 1.16245 | 6.97470 |
| 78 | 2 | 5 | 4 | 5 | 0.71397 | 3.56985 |
| 79 | 2 | 5 | 5 | 6 | 1.40754 | 8.44524 |
| 80 | 2 | 5 | 6 | 2 | 1.35376 | 2.70752 |
| 81 | 2 | 5 | 7 | 3 | 1.31798 | 3.95394 |
| 82 | 2 | 5 | 8 | 2 | 1.13548 | 2.27096 |
| 83 | 2 | 5 | 9 | 6 | 1.33278 | 7.99668 |
| 84 | 2 | 6 | 1 | 1 | 1.25496 | 1.25496 |
| 85 | 2 | 6 | 3 | 3 | 1.10968 | 3.32904 |
| 86 | 2 | 6 | 4 | 1 | 0.66120 | 0.66120 |
| 87 | 2 | 6 | 5 | 2 | 1.35477 | 2.70954 |

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|---|---|---|---|---|---|---|
| 88 | 2 | 6 | 6 | 2 | 1.30099 | 2.60198 |
| 89 | 2 | 6 | 7 | 2 | 1.26521 | 2.53042 |
| 90 | 2 | 6 | 8 | 2 | 1.08272 | 2.16544 |
| 91 | 2 | 6 | 9 | 2 | 1.28002 | 2.56004 |
| 92 | 2 | 7 | 8 | 1 | 1.23830 | 1.23830 |
| 93 | 2 | 7 | 9 | 5 | 1.43560 | 7.17800 |
| 94 | 2 | 8 | 2 | 1 | 2.58563 | 2.58563 |
| 95 | 2 | 8 | 3 | 4 | 1.70283 | 6.81132 |
| 96 | 2 | 8 | 4 | 1 | 1.25435 | 1.25435 |
| 97 | 2 | 8 | 5 | 1 | 1.94792 | 1.94792 |
| 98 | 2 | 8 | 6 | 2 | 1.89414 | 3.78828 |
| 99 | 2 | 8 | 7 | 2 | 1.85836 | 3.71672 |
| 100 | 2 | 8 | 8 | 1 | 1.67587 | 1.67587 |
| 101 | 2 | 8 | 9 | 2 | 1.87317 | 3.74634 |
| 102 | 2 | 9 | 3 | 1 | 2.65918 | 2.65918 |
| 103 | 2 | 9 | 4 | 2 | 2.21070 | 4.42140 |
| 104 | 2 | 10 | 3 | 1 | 5.38320 | 5.38320 |
| 105 | 3 | 1 | 4 | 1 | 5.92843 | 5.92843 |
| 106 | 3 | 2 | 1 | 1 | 2.37462 | 2.37462 |
| 107 | 3 | 2 | 2 | 3 | 3.11213 | 9.33639 |
| 108 | 3 | 2 | 5 | 2 | 2.47443 | 4.94886 |
| 109 | 3 | 2 | 6 | 1 | 2.42065 | 2.42065 |
| 110 | 3 | 2 | 8 | 1 | 2.20238 | 2.20238 |
| 111 | 3 | 2 | 9 | 1 | 2.39967 | 2.39967 |
| 112 | 3 | 3 | 1 | 3 | 1.02623 | 3.07869 |
| 113 | 3 | 3 | 2 | 5 | 1.76375 | 8.81875 |
| 114 | 3 | 3 | 3 | 16 | 0.88095 | 14.09520 |
| 115 | 3 | 3 | 4 | 7 | 0.43247 | 3.02729 |
| 116 | 3 | 3 | 5 | 9 | 1.12604 | 10.13436 |
| 117 | 3 | 3 | 6 | 5 | 1.07226 | 5.36130 |
| 118 | 3 | 3 | 7 | 7 | 1.03648 | 7.25536 |
| 119 | 3 | 3 | 8 | 6 | 0.85399 | 5.12394 |
| 120 | 3 | 3 | 9 | 11 | 1.05128 | 11.56408 |
| 121 | 3 | 4 | 1 | 3 | 0.71852 | 2.15556 |
| 122 | 3 | 4 | 2 | 5 | 1.45603 | 7.28015 |
| 123 | 3 | 4 | 3 | 15 | 0.57324 | 8.59860 |
| 124 | 3 | 4 | 4 | 13 | 0.12476 | 1.62188 |
| 125 | 3 | 4 | 5 | 14 | 0.81833 | 11.45662 |
| 126 | 3 | 4 | 6 | 3 | 0.76455 | 2.29365 |
| 127 | 3 | 4 | 7 | 4 | 0.72877 | 2.91508 |
| 128 | 3 | 4 | 8 | 8 | 0.54628 | 4.37024 |
| 129 | 3 | 4 | 9 | 5 | 0.74357 | 3.71785 |
| 130 | 3 | 5 | 1 | 2 | 0.68975 | 1.37950 |
| 131 | 3 | 5 | 2 | 12 | 1.42727 | 17.12724 |
| 132 | 3 | 5 | 3 | 16 | 0.54447 | 8.71152 |
| 133 | 3 | 5 | 4 | 12 | 0.09600 | 1.15200 |

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| 134 | 3 | 5 | 5 | 14 | 0.78956 | 11.05384 |
| 135 | 3 | 5 | 6 | 3 | 0.73578 | 2.20734 |
| 136 | 3 | 5 | 7 | 9 | 0.70001 | 6.30009 |
| 137 | 3 | 5 | 8 | 3 | 0.51751 | 1.55253 |
| 138 | 3 | 5 | 9 | 8 | 0.71481 | 5.71848 |
| 139 | 3 | 6 | 1 | 1 | 0.63699 | 0.63699 |
| 140 | 3 | 6 | 2 | 2 | 1.37450 | 2.74900 |
| 141 | 3 | 6 | 3 | 5 | 0.49171 | 2.45855 |
| 142 | 3 | 6 | 4 | 4 | 0.04323 | 0.17292 |
| 143 | 3 | 6 | 5 | 7 | 0.73680 | 5.15760 |
| 144 | 3 | 6 | 6 | 2 | 0.68302 | 1.36604 |
| 145 | 3 | 6 | 7 | 2 | 0.64724 | 1.29448 |
| 146 | 3 | 6 | 8 | 2 | 0.46475 | 0.92950 |
| 147 | 3 | 6 | 9 | 2 | 0.66204 | 1.32408 |
| 148 | 3 | 7 | 1 | 2 | 0.79257 | 1.58514 |
| 149 | 3 | 7 | 2 | 3 | 1.53009 | 4.59027 |
| 150 | 3 | 7 | 3 | 9 | 0.64729 | 5.82561 |
| 151 | 3 | 7 | 4 | 5 | 0.19881 | 0.99405 |
| 152 | 3 | 7 | 5 | 2 | 0.89238 | 1.78476 |
| 153 | 3 | 7 | 6 | 2 | 0.83860 | 1.67720 |
| 154 | 3 | 7 | 8 | 4 | 0.62033 | 2.48132 |
| 155 | 3 | 7 | 9 | 2 | 0.81763 | 1.63526 |
| 156 | 3 | 8 | 1 | 1 | 1.23014 | 1.23014 |
| 157 | 3 | 8 | 2 | 1 | 1.96765 | 1.96765 |
| 158 | 3 | 8 | 3 | 2 | 1.08486 | 2.16972 |
| 159 | 3 | 8 | 4 | 3 | 0.63638 | 1.90914 |
| 160 | 3 | 8 | 5 | 1 | 1.32995 | 1.32995 |
| 161 | 3 | 8 | 8 | 2 | 1.05790 | 2.11580 |
| 162 | 3 | 8 | 9 | 2 | 1.25519 | 2.51038 |
| 163 | 3 | 9 | 2 | 1 | 2.92400 | 2.92400 |
| 164 | 3 | 9 | 3 | 1 | 2.04120 | 2.04120 |
| 165 | 3 | 9 | 5 | 1 | 2.28630 | 2.28630 |
| 166 | 3 | 9 | 8 | 1 | 2.01424 | 2.01424 |
| 167 | 3 | 9 | 9 | 3 | 2.21154 | 6.63462 |
| 168 | 3 | 10 | 4 | 1 | 4.31675 | 4.31675 |
| 169 | 3 | 10 | 5 | 1 | 5.01032 | 5.01032 |
| 170 | 4 | 1 | 2 | 1 | 7.06970 | 7.06970 |
| 171 | 4 | 2 | 3 | 1 | 2.03933 | 2.03933 |
| 172 | 4 | 2 | 5 | 1 | 2.28442 | 2.28442 |
| 173 | 4 | 2 | 7 | 1 | 2.19486 | 2.19486 |
| 174 | 4 | 3 | 1 | 5 | 0.83622 | 4.18110 |
| 175 | 4 | 3 | 2 | 2 | 1.57374 | 3.14748 |
| 176 | 4 | 3 | 3 | 4 | 0.69094 | 2.76376 |
| 177 | 4 | 3 | 4 | 7 | 0.24247 | 1.69729 |
| 178 | 4 | 3 | 5 | 7 | 0.93604 | 6.55228 |
| 179 | 4 | 3 | 6 | 1 | 0.88226 | 0.88226 |

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| 180 | 4 | 3 | 7 | 3 | 0.84648 | 2.53944 |
| 181 | 4 | 3 | 8 | 2 | 0.66398 | 1.32796 |
| 182 | 4 | 3 | 9 | 10 | 0.86128 | 8.61280 |
| 183 | 4 | 4 | 1 | 3 | 0.52851 | 1.58553 |
| 184 | 4 | 4 | 2 | 8 | 1.26603 | 10.12824 |
| 185 | 4 | 4 | 3 | 14 | 0.38323 | 5.36522 |
| 186 | 4 | 4 | 4 | 5 | 0.00100 | 0.00500 |
| 187 | 4 | 4 | 5 | 8 | 0.62832 | 5.02656 |
| 188 | 4 | 4 | 6 | 2 | 0.57454 | 1.14908 |
| 189 | 4 | 4 | 7 | 7 | 0.53876 | 3.77132 |
| 190 | 4 | 4 | 8 | 8 | 0.35627 | 2.85016 |
| 191 | 4 | 4 | 9 | 9 | 0.55357 | 4.98213 |
| 192 | 4 | 5 | 1 | 3 | 0.49975 | 1.49925 |
| 193 | 4 | 5 | 2 | 4 | 1.23726 | 4.94904 |
| 194 | 4 | 5 | 3 | 17 | 0.35447 | 6.02599 |
| 195 | 4 | 5 | 4 | 7 | 0.00100 | 0.00700 |
| 196 | 4 | 5 | 5 | 11 | 0.59956 | 6.59516 |
| 197 | 4 | 5 | 6 | 2 | 0.54578 | 1.09156 |
| 198 | 4 | 5 | 7 | 6 | 0.51000 | 3.06000 |
| 199 | 4 | 5 | 8 | 3 | 0.32751 | 0.98253 |
| 200 | 4 | 5 | 9 | 6 | 0.52480 | 3.14880 |
| 201 | 4 | 6 | 1 | 2 | 0.44698 | 0.89396 |
| 202 | 4 | 6 | 3 | 7 | 0.30170 | 2.11190 |
| 203 | 4 | 6 | 5 | 3 | 0.54679 | 1.64037 |
| 204 | 4 | 6 | 7 | 4 | 0.45723 | 1.82892 |
| 205 | 4 | 6 | 8 | 4 | 0.27474 | 1.09896 |
| 206 | 4 | 6 | 9 | 4 | 0.47204 | 1.88816 |
| 207 | 4 | 7 | 3 | 2 | 0.45729 | 0.91458 |
| 208 | 4 | 7 | 4 | 2 | 0.00881 | 0.01762 |
| 209 | 4 | 7 | 5 | 1 | 0.70238 | 0.70238 |
| 210 | 4 | 7 | 9 | 1 | 0.62762 | 0.62762 |
| 211 | 4 | 8 | 1 | 1 | 1.04013 | 1.04013 |
| 212 | 4 | 8 | 2 | 1 | 1.77765 | 1.77765 |
| 213 | 4 | 8 | 3 | 2 | 0.89485 | 1.78970 |
| 214 | 4 | 8 | 4 | 2 | 0.44637 | 0.89274 |
| 215 | 4 | 8 | 5 | 8 | 1.13994 | 9.11952 |
| 216 | 4 | 8 | 6 | 2 | 1.08616 | 2.17232 |
| 217 | 4 | 8 | 7 | 1 | 1.05038 | 1.05038 |
| 218 | 4 | 8 | 8 | 2 | 0.86789 | 1.73578 |
| 219 | 4 | 8 | 9 | 2 | 1.06519 | 2.13038 |
| 220 | 4 | 9 | 4 | 1 | 1.40272 | 1.40272 |
| 221 | 4 | 9 | 6 | 1 | 2.04251 | 2.04251 |
| 222 | 5 | 1 | 3 | 1 | 6.31875 | 6.31875 |
| 223 | 5 | 1 | 4 | 1 | 5.87028 | 5.87028 |
| 224 | 5 | 1 | 7 | 1 | 6.47429 | 6.47429 |
| 225 | 5 | 2 | 1 | 1 | 2.31647 | 2.31647 |

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|---|---|---|---|---|---|---|
| 226 | 5 | 2 | 7 | 1 | 2.32672 | 2.32672 |
| 227 | 5 | 2 | 9 | 1 | 2.34152 | 2.34152 |
| 228 | 5 | 3 | 1 | 2 | 0.96808 | 1.93616 |
| 229 | 5 | 3 | 3 | 5 | 0.82280 | 4.11400 |
| 230 | 5 | 3 | 4 | 4 | 0.37432 | 1.49728 |
| 231 | 5 | 3 | 5 | 3 | 1.06789 | 3.20367 |
| 232 | 5 | 3 | 6 | 1 | 1.01411 | 1.01411 |
| 233 | 5 | 3 | 7 | 6 | 0.97833 | 5.86998 |
| 234 | 5 | 3 | 8 | 3 | 0.79584 | 2.38752 |
| 235 | 5 | 3 | 9 | 6 | 0.99313 | 5.95878 |
| 236 | 5 | 4 | 1 | 3 | 0.66037 | 1.98111 |
| 237 | 5 | 4 | 2 | 4 | 1.39788 | 5.59152 |
| 238 | 5 | 4 | 3 | 5 | 0.51509 | 2.57545 |
| 239 | 5 | 4 | 4 | 6 | 0.06661 | 0.39966 |
| 240 | 5 | 4 | 5 | 11 | 0.76018 | 8.36198 |
| 241 | 5 | 4 | 6 | 4 | 0.70640 | 2.82560 |
| 242 | 5 | 4 | 7 | 10 | 0.67062 | 6.70620 |
| 243 | 5 | 4 | 8 | 5 | 0.48812 | 2.44060 |
| 244 | 5 | 4 | 9 | 14 | 0.68542 | 9.59588 |
| 245 | 5 | 5 | 1 | 4 | 0.63160 | 2.52640 |
| 246 | 5 | 5 | 2 | 4 | 1.36912 | 5.47648 |
| 247 | 5 | 5 | 3 | 15 | 0.48632 | 7.29480 |
| 248 | 5 | 5 | 4 | 4 | 0.03784 | 0.15136 |
| 249 | 5 | 5 | 5 | 11 | 0.73141 | 8.04551 |
| 250 | 5 | 5 | 6 | 5 | 0.67763 | 3.38815 |
| 251 | 5 | 5 | 7 | 7 | 0.64185 | 4.49295 |
| 252 | 5 | 5 | 8 | 2 | 0.45936 | 0.91872 |
| 253 | 5 | 5 | 9 | 6 | 0.65666 | 3.93996 |
| 254 | 5 | 6 | 2 | 3 | 1.31635 | 3.94905 |
| 255 | 5 | 6 | 3 | 1 | 0.43356 | 0.43356 |
| 256 | 5 | 6 | 4 | 1 | 0.00100 | 0.00100 |
| 257 | 5 | 6 | 5 | 5 | 0.67865 | 3.39325 |
| 258 | 5 | 6 | 6 | 1 | 0.62487 | 0.62487 |
| 259 | 5 | 6 | 7 | 1 | 0.58909 | 0.58909 |
| 260 | 5 | 6 | 8 | 2 | 0.40660 | 0.81320 |
| 261 | 5 | 6 | 9 | 2 | 0.60389 | 1.20778 |
| 262 | 5 | 7 | 1 | 4 | 0.73442 | 2.93768 |
| 263 | 5 | 7 | 4 | 1 | 0.14066 | 0.14066 |
| 264 | 5 | 7 | 7 | 5 | 0.74467 | 3.72335 |
| 265 | 5 | 7 | 8 | 2 | 0.56218 | 1.12436 |
| 266 | 5 | 7 | 9 | 4 | 0.75948 | 3.03792 |
| 267 | 5 | 8 | 1 | 1 | 1.17199 | 1.17199 |
| 268 | 5 | 8 | 5 | 1 | 1.27180 | 1.27180 |
| 269 | 5 | 8 | 7 | 1 | 1.18224 | 1.18224 |
| 270 | 5 | 8 | 9 | 4 | 1.19704 | 4.78816 |
| 271 | 5 | 9 | 1 | 1 | 2.12833 | 2.12833 |

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|---|---|---|---|---|---|---|
| 272 | 5 | 9 | 4 | 1 | 1.53458 | 1.53458 |
| 273 | 5 | 9 | 6 | 1 | 2.17437 | 2.17437 |
| 274 | 5 | 9 | 9 | 1 | 2.15339 | 2.15339 |
| 275 | 5 | 10 | 2 | 1 | 5.58987 | 5.58987 |
| | | | | 1000 | | 1001.01 |